Semester project in mathematics, BA5

Cayley Graphs and growth of groups.

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Autumn 2008

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CHAPTER 1

Introduction

One of the most fundamental objects in algebra is the group. There exist plenty of different groups. In the following article we consider some free abelian groups, cyclic groups, free groups and fundamental groups as well as the products of groups and draw picture of them.

In the first section, see Section 2.1, we define the CAYLEY graph, a way of representing a group given its set of generators. Some examples of the groups we mentioned further are exhibited and we draw their CAYLEY graphs. These pictures will indicate that the graphs are of different types in the sense that some are finite and some growing faster than the others.

Then a way to distinguish the different types of CAYLEY graphs is given, namely considering the growth of the graphs. In Section **??**, we define a natural distance on those CAYLEY graphs. Then we look at the number of elements in the ball of radius n and finally we define the growth as a function of this number of elements. We state that the number of element in the ball of radius n can be a polynomial, a mixed or an exponential expression which will give different type of growth.
As it is not always easy to exhibit the number of elements in the ball of radius n we introduce some way of calculating it by considering the different types of words in the group and their length. We show this method on some finitely generated groups, see Examples **??** and **??**, and draw their CAYLEY graphs to figure the result one obtained theoretically. We also show that if a group has exponential growth it is independent of the generating set when this one is finite.

After that we separately consider in Section **??** the fundamental groups of connected topological spaces. A brief recall of some topological definitions and results is done in order to define the fundamental group of a topological space. Then we determine the fundamental group of a contractible space and the one of the sphere of dimension two.

Then we enunciate the theorem of SEIFERT and VAN KAMPEN.

The theorem of SEIFERT and VAN KAMPEN, see Theorem **??**, is an important result that helps to determine the fundamental group of topological spaces. Examples of finding the fundamental group trough the help of the VAN KAMPEN's theorem are given in Section **??**.

To finish, we look in the last section, see Section **??**, at the growth of the fundamental groups that one has computed. It is not always possible to look at the number of element in the ball of radius n, especially when the group has a lot of generators. Hence we try to determine the type of growth of the group in another way, namely if one can find a subset in the set of generators that generates a free group, then the group has exponential growth.

Theory

2.1 Cayley Graphs

Definition 1 (Group presentation)**.** A *presentation* of a group G is a pair $(\{s_j\}_{j\in J}, \{r_i\}_{i\in I})$ where $\{s_j\}_{j\in J}$ is a basis for a free group F such that there exists an homomorphism $\pi : F \to G$ and $\{r_i\}_{i\in I}$ is a family in F which generates $\{x \in F \mid \pi(x) = 0\}$ as a normal subgroup.

One denotes it by $\gamma =< S \mid (r_i)_{i\in I} >$.

According to this definition one can present the Cartesian product of G and H as $G \times H = < (S_1, 1), (1, S_2) \mid R_1, R_2, ghg^{-1}h^{-1} = 1 >$ and in the same way, one defines the free product of G and H as $G * H =< S_1 \cup S_2 \mid R_1, R_2 >$ where $G =< S_1 \mid R_1 >$ and $H =< S_2 \mid R_2 >$.

Definition 2 (Product with amalgamation)**.** Let A, Γ_1, Γ_2 be three groups and let ι_j be the two applications $A \to \Gamma_j$ for $j = 1, 2$. The corresponding *free product* of Γ_1 and Γ_2 *with amalgamation* over A is denoted by $\Gamma_1 *_A \Gamma_2$.

One can characterize $\Gamma_1 *_A \Gamma_2$ as $\Gamma_1 * \Gamma_2/N$ where N is the smallest normal subgroup of $\Gamma_1 * \Gamma_2$ containning all of the elements that fulfil $\iota_1(a) = \iota_1(a)$ for all $a \in A$.

The free product $*$ is a particular case of product with amalgamation, that is, when $A = \{id\}$ in the previous definition.

In terms of group presentation, if $\Gamma_1 =< S_1 \mid R_1 >$ and $\Gamma_2 =< S_2 \mid R_2 >$ then $\Gamma_1 *_A \Gamma_2 =< S_1 \cup S_2 \mid R_1, R_2, \iota_1(a) = \iota_2(a), \forall a \in A >$.

Then one has the following universal property.

- For a group G and two homomorphisms $h_j \colon \Gamma_j \to G$ such that $h_1(a) = h_2(a)$ for all $a \in A$ there exists a unique isomorphism $h \colon \Gamma_1 *_A \Gamma_2 \to G$ such that $h(y) = h_j(y_j)$ for all $y_j \in \Gamma_j$. Therefore $\Gamma_1 *_A \Gamma_2 \cong G$.

Example 3. *It can be shown that* $SL(2, \mathbb{Z}) \cong (\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z})$ *with* $s_1 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ *generating* $\mathbb{Z}/6\mathbb{Z}$ *and* $s_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *generating* $\mathbb{Z}/4\mathbb{Z}$. *In par-*

ticular, $\iota_i = id$ and $SL(2,\mathbb{Z})$ has the presentation $SL(2,\mathbb{Z}) = (\mathbb{Z}/6\mathbb{Z}) *_{\mathbb{Z}/2\mathbb{Z}} (\mathbb{Z}/4\mathbb{Z}) =< s_1, s_2 \mid s_1^6 = id, s_2^4 = id, \iota_1(s_1^3) = \iota_2(s_2^2) >$.

Let us now define a Cayley graph.

Definition 4 (Cayley graph)**.** The *Cayley graph* of a group G is the graph wherein every elements of G is a vertex and the wedges represent the elements of $\{(g, gs) \mid g \in G, s \in S \cup S^{-1}\}$ where S is generating G. One uses the notation $Cay(G,S)$.

Let us consider some examples of groups and illustrate their Cayley graphs.

Example 5.

- *The free abelian groups with the canonical generators as $\mathbb{Z}$, $\mathbb{Z}^2$, ..., $\mathbb{Z}^k$.*

The graph of $\mathbb{Z}$ is the real line with the integer number as vertex, see Figure **??**.

Figure 2.1: $\mathbb{Z}$

One can write $\mathbb{Z}^2$ as $\mathbb{Z}^2 =< a, b \mid aba^{-1}b^{-1} >$ and the Cayley graph is shown in Figure **??**.

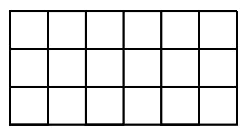


Figure 2.2: $\mathbb{Z}^2$

For $\mathbb{Z}^k$ with the following presentation $\mathbb{Z}^k =< s_1, \ldots, s_k \mid s_i s_j s_i^{-1} s_j^{-1} = 1, \forall i \neq j >$, corresponding to the one with the canonical generators, the Cayley graph will be a grid in $\mathbb{R}^k$.

- *The free groups with canonical generators.*

Let us first look at the Cayley graph of $F_2 =< a, b >$, see Figure **??**.

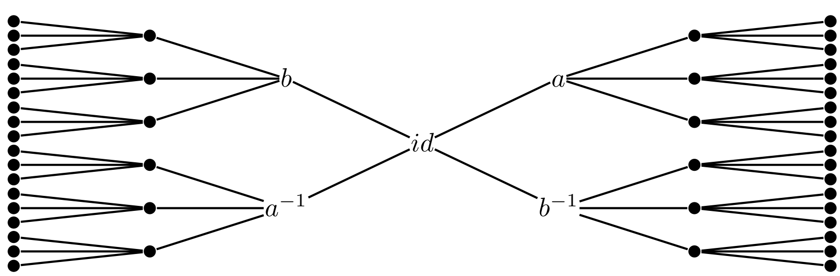


Figure 2.3: F_2.

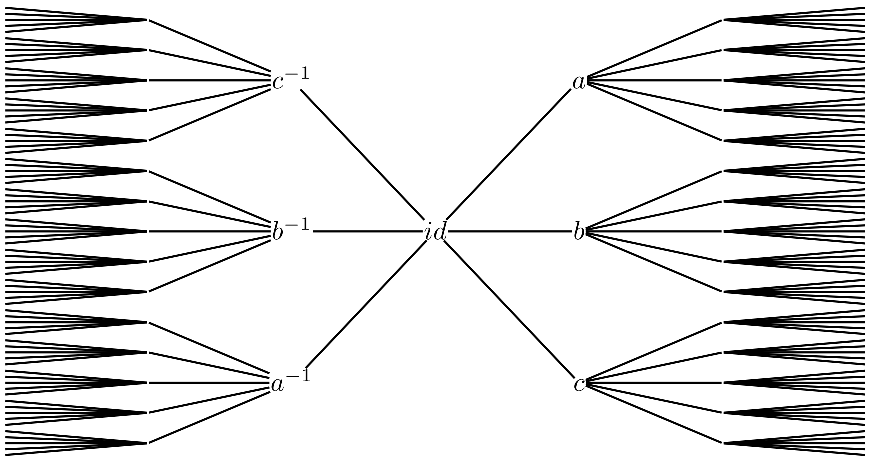


Figure 2.4: F_3.

*With the graph of F_3, see Figure **??**, one sees that the free groups have a fastest growth than the free abelian ones.*

One writes $\mathbb{F}_k$ as $F_k =< s_1, \ldots, s_k >$. The Cayley graph is a tree with $2k$ wedges at the first level and then $2k-1$ at each level.

- *The cyclic groups with canonical generators.*

*One writes those groups as $\mathbb{Z}/m\mathbb{Z} =< a \mid a^m = 1 >$ and the Cayley graph is a m-gone. Figure **??** represents the group $\mathbb{Z}/5\mathbb{Z} =< a \mid a^5 = 1 >$.*

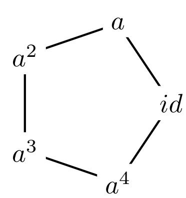


Figure 2.5: $\mathbb{Z}/5\mathbb{Z}$.

- *The free products of the form $\mathbb{Z}/m\mathbb{Z} * \mathbb{Z}/n\mathbb{Z}$ with canonical generators.*

*Figure **??** is the Cayley graph of $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} =< a, b \mid a^2 = 1, b^2 = 1 >$.*

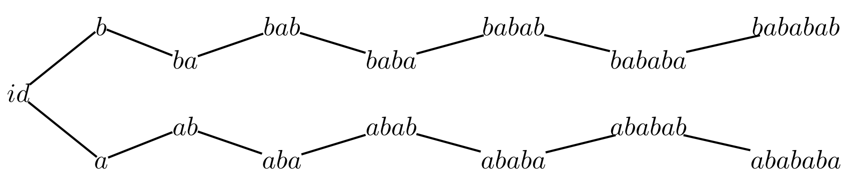


Figure 2.6: $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$

One recognize in the graph of Figure **??** *that the group has a* $\mathbb{Z}/2\mathbb{Z}$ *component as from a word ending with* a *there is always only one vertex and one also sees that the second component is* $\mathbb{Z}/2\mathbb{Z}$ *since the same holds from the words in* b. *Figure* **??** *represents the Cayley graph of* $\mathbb{Z}/2\mathbb{Z}*\mathbb{Z}/4\mathbb{Z} =< a, b \mid a^2 = 1, b^4 = 1 >$.

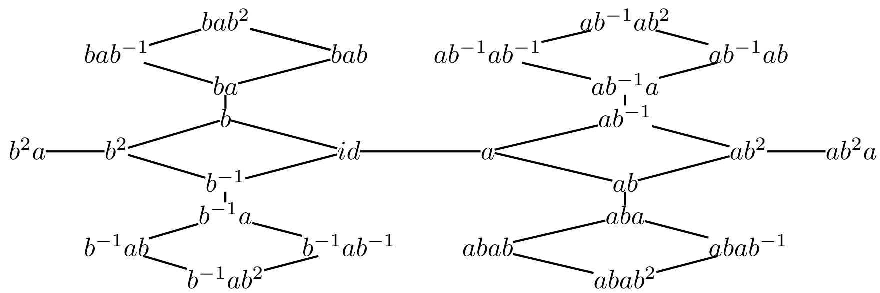


Figure 2.7: $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/4\mathbb{Z}$

In the same way, one sees in Figure **??** *that we have a* $\mathbb{Z}/2\mathbb{Z}$ *component as there is always only one vertex leading to an* a*-ending word. Analogously the* $\mathbb{Z}/4\mathbb{Z}$ *component can be seen in the diamond looking part of the figure.*

The Cayley graph of a group can be considerably different depending on the set of generators, as can seen in the following examples.

- $\mathbb{Z}$ with different generators.

  Let us first look again at the graph of $\mathbb{Z}$ with the canonical generators.

Figure 2.8: $\mathbb{Z}$

  In a second time one considers $\mathbb{Z}$ with the set of genertors $\{3, 4\}$.

Figure 2.9: $\mathbb{Z} =< 3, 4 >$.

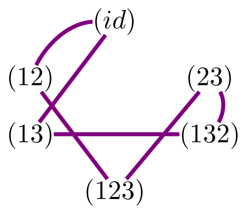
  One sees that both graphs are different, but from far away they tends to look like a line, that is, to be the same.

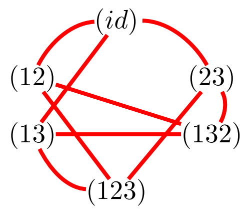
- The symmetric group $G = Sym(3)$.

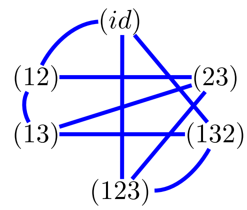
  This example is different from the previous ones as the Cayley graph of $Sym(3)$ will be closed, since $Sym(3)$ is a finite group. But one can still observe a change depending on the generating set.

  Let us consider the following sets of generators

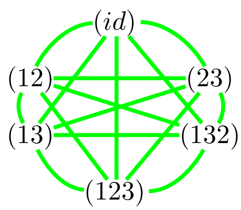
$S_1 = \{(12), (13)\}$, $S_2 = \{(12), (23), (13)\}$, $S_3 = \{(12), (123)\}$ and







$S_4 = Sym(3) \setminus \{id\}$.



2.2 Growth of groups

In order to qualify the differences that one remarked in these examples, one wants to define a distance on the Cayley graphs and measure their growths.

Definition 6 (Distance on G)**.** One defines

$$\begin{aligned} l_S : \quad G &\longrightarrow \mathbb{Z} \\ g &\longmapsto \min\{k \;\mid\; g = s_1 \cdots s_k, s_i \in S \cup S^{-1}\}, \end{aligned}$$

as the *length* of an element in G with respect to the generating set S and

$$\begin{aligned} d_S : \quad G \times G &\longrightarrow \mathbb{R} \\ (g_1, g_2) &\longmapsto l_S(id, g_1^{-1} g_2) \end{aligned}$$

as the *distance on* G.

Claim 7. *d_S is a distance.*

Proof.

Positivity : $d_S(x, y) = l_S(id, x^{-1}y) = \min\{k \mid x^{-1}y = s_1 \cdots s_k, s_i \in S \cup S^{-1}\} \geq 0$; the equality holds if and only if $x = y$. Indeed by convention the element generated by an empty product is the identity, so that x^{-1} the inverse of y and $x = y$.

Symmetry :

$$\begin{aligned} d_S(x, y) &= l_S(id, x^{-1}y) = \min\{k \mid x^{-1}y = s_1 \cdots s_k, s_i \in S \cup S^{-1}\} \\ &= \min\{k \mid y^{-1}x = s_k^{-1} \cdots s_1^{-1}, s_i \in S \cup S^{-1}\} = d_S(y, x), \end{aligned}$$

as $s_i \in S \cup S^{-1}$ if and only if $s_i^{-1} \in S \cup S^{-1}$.

Triangle Inequality : Let x, y and z be such that $d_S(x,y) = k$ and $d_S(y,z) = l$. In particular there exist $s_1, \ldots, s_k, r_1, \ldots, r_l \in S \cup S^{-1}$ such that $x^{-1}y = s_1 \cdots s_k$ and $y^{-1}z = r_1 \cdots r_l$. From $x^{-1}z = (x^{-1}y)(y^{-1}z) = (s_1 \cdots s_k)(r_1 \cdots r_l)$, one obtains that

$$d_S(x,z) \leq k + l = d_S(x,y) + d_S(y,z).$$

□

Observe that d_S is the natural distance on the Cayley graph $Cay(G,S)$.

Definition 8 (Ball on Cayley graph)**.** The *ball*

$$B_n(G,S) = \{g \in G \mid d_S(id, g) \leq n\}$$

is the set of points with a distance to the identity lower or equal than n, that is, the elements generated by at most n generators.

One denotes by $\beta_n(G,S) = \mid B_n(G,S) \mid$ the number of elements containned in the ball with radius n.

Definition 9 (Sphere on Cayley graph)**.** The *sphere*

$$C_n(G,S) = B_n(G,S) \backslash B_{n-1}(G,S) = \{g \in G \mid d_S(id,g) = n\}$$

is the set of points on Cayley graph with distance n to the identity, that is, the elements whose minimal number of generators is exactly n.

One denotes by $\sigma_n(G,S) = \mid C_n(G,S) \mid$, the number of elements with length n.

Observe that the following relations are direct consequences of the previous definitions

- $\beta_n = \sigma_0 + \sigma_1 + \cdots + \sigma_n = \displaystyle\sum_{i=0}^{n} \sigma_i$,
- $\sigma_n = \beta_n - \beta_{n-1}$.

Let us now look at some examples and calculate σ_n and β_n.

Example 10. *1.* $\mathbb{Z}$, $S = \{1\}$, *then* $B_n = \{-n, -(n-1), \ldots, 0, \ldots, n-1, n\}$ *and* $C_n = \{-n, n\}$, *hence* $\beta_n = 2n + 1$, $\sigma_n = 2$.

2. $\mathbb{Z}^k$, $S = \{e_1, \ldots, e_n\}$, *by definition one has that*

$C_n = \{(n_1, \ldots, n_k) \mid |n_1| + \cdots + |n_k| = n\}$, which gives

$$\begin{aligned}
\sigma_n &= \sum_{i=1}^{n} 2\#\{(i, n_2, \ldots, n_k) \mid |i| + |n_2| + \cdots + |n_k| = n\} \\
&+ \#\{(0, n_2, \ldots, n_k) \mid |n_2| + \cdots + |n_k| = n\} \\
&= 2\sum_{i=1}^{n} \#\{(i, n_2, \ldots, n_k) \mid |n_2| + \cdots + |n_k| = n - i\} + \sigma_n(\mathbb{Z}^{k-1}) \\
&= 2\sum_{i=1}^{n} \sigma_{n-i}(\mathbb{Z}^{k-1}) + \sigma_n(\mathbb{Z}^{k-1}) = 2\beta_{n-1}(\mathbb{Z}^{k-1}) + \sigma_n(\mathbb{Z}^{k-1}) \\
&= \beta_{n-1}(\mathbb{Z}^{k-1}) + \beta_n(\mathbb{Z}^{k-1}) \text{ and} \\
\beta_n &= \beta_0 + \sum_{j=1}^{n} (\beta_{j-1}(\mathbb{Z}^{k-1}) + \beta_j(\mathbb{Z}^{k-1})) \\
&= (\beta_0 + \beta_0 + \beta_1 + \beta_1 + \beta_2 + \cdots + \beta_{n-1} + \beta_n)(\mathbb{Z}^{k-1}) \\
&= 2\sum_{j=0}^{n-1} \beta_j(\mathbb{Z}^{k-1}) + \beta_n(\mathbb{Z}^{k-1}).
\end{aligned}$$

It follows from the former that the number of elements in $C_n(\mathbb{Z}^k)$, $\sigma_n(\mathbb{Z}^k)$, is a polynomial of degree $k-1$ and the number of the ones in $B_n(\mathbb{Z}^k)$, $\beta_n(\mathbb{Z}^k)$, is a polynomial of degree k. Hence

for $\mathbb{Z}^2$ with $S = \{(1,0), (0,1)\}$, one has that $\sigma_0 = 1$, $\sigma_1 = 4$, …, $\sigma_n = 4n$ thus

$$\beta_n = \sum_{k=0}^{n} \sigma_k = 1 + \sum_{i=1}^{n} 4i = 1 + 4\frac{(n+1)n}{2} = 2n^2 + 2n + 1.$$

and for $\mathbb{Z}^3$ with $S = \{(1,0,0), (0,1,0), (0,0,1)\}$, the general formula implies that $\sigma_n = 4n^2 + 2$, for all $n \geq 1$ and then

$$\begin{aligned}
\beta_n &= 1 + \sum_{i=1}^{n} \sigma_n = 1 + \sum_{i=1}^{n} (4i^2 + 2) \\
&= 1 + 4\sum_{i=1}^{n} i^2 + 2n = 1 + 4\frac{n(n+1)(2n+1)}{6} + 2n \\
&= \frac{4n^3}{3} + 2n^2 + \frac{8n}{3} + 1.
\end{aligned}$$

3. *F_2, $S = \{a, b\}$, then $\sigma_0 = 1$, $\sigma_1 = 4$, $\sigma_2 = 4 \cdot 3$ and $\sigma_n = 4 \cdot 3^{n-1}$. It follows that $\beta_0 = 1$, $\beta_1 = 1 + 4$, $\beta_2 = 1 + 4 + 3 \cdot 4$, and by induction*

$$\beta_n = 1 + 4\sum_{i=0}^{n-1} 3^i = 1 + 4\left(\frac{3^n - 1}{3 - 1}\right) = 1 + 2(3^n - 1) = 2 \cdot 3^n - 1.$$

4. *F_k, $S = \{a_1, \ldots, a_k\}$, so $\beta_0 = 1$, $\beta_1 = 1 + 2k$, $\beta_2 = 1 + 2k + 2k(2k-1)$, which gives*

$$\begin{aligned}
\beta_n &= 1 + 2k\sum_{i=0}^{n-1}(2k-1)^i = 1 + 2k\left(\frac{(2k-1)^n - 1}{2k-1-1}\right) \\
&= 1 + \frac{k((2k-1)^n - 1)}{k-1} = \frac{k - 1 + k(2k-1)^n - k}{k-1} \\
&= \frac{k(2k-1)^n - 1}{k-1}.
\end{aligned}$$

5. *$\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} =< a, b \mid a^2 = 1, b^2 = 1 >$, then $\beta_0 = 1 = \sigma_0$ and $\sigma_n = 2$, for all $n \geq 1$, so*

$$\beta_n = \sigma_0 + \sum_{i=1}^{n}\sigma_n = 1 + \sum_{i=1}^{n} 2 = 1 + 2n.$$

As in the free groups one has no relations, the free groups have the highest values of σ_n and β_n and therefore σ_n and β_n are bounded, which is stated in the following lemmas.

Lemma 11. *For all groups G generated by some finite set S with card$(S) = k$,*

$$\sigma_n(G, S) \leq (2k)(2k-1)^{n-1}.$$

Proof. As some inverses may already be containned in S, if there are k elements in S, there could be at most $2k$ in $S \cup S^{-1}$. Thus $\sigma_1 \leq 2k$.

The elements one counts in σ_2 can be written as an element s_1 of C_1 composed with an element s_2 from $S \cup S^{-1}$. There are at most $2k-1$ such possible s_2, because there are at most $2k$ elements in $S \cup S^{-1}$ and s_2 can not be the inverse of s_1, so one must exclude it. This implies that $\sigma_2 \leq 2k(2k-1)$. Analogously, $\sigma_n \leq \sigma_{n-1}(2k-1) \leq \cdots \leq 2k(2k-1)^{n-1}$. The result is reached. □

Lemma 12. *For all groups G generated by some finite set S with card$(S) = k$,*

$$1 \leq \beta_n(G, S) \leq \frac{k(2k-1)^n - 1}{k-1}.$$

Proof. On the one hand, as $\beta_0(G, S) = 1$ and $B_0(G, S) \subset B_n(G, S)$ for all n one has $1 \leq \beta_n(G, S)$. On the other hand, one knows that $\sigma_n \leq (2k)(2k-1)^{n-1}$ by Lemma **??** and $\beta_n = \sum_{i=0}^{n}\sigma_n$, so

$$\begin{aligned}
\beta_n &= \sum_{i=0}^{n}\sigma_n = \sigma_0 + \sum_{i=1}^{n}\sigma_n \\
&\leq 1 + \sum_{i=1}^{n}(2k)(2k-1)^{i-1} = 1 + 2k\sum_{i=1}^{n}(2k-1)^{i-1} = 1 + 2k\sum_{i=0}^{n}(2k-1)^i \\
&= 1 + 2k\frac{(2k-1)^n - 1}{(2k-1) - 1} = 1 + k\frac{(2k-1)^n - 1}{k-1} = \frac{k(2k-1)^n - 1}{k-1}.
\end{aligned}$$

□

With these examples one could have seen that β can be of different forms. It may be

- a polynomial expression of type $\beta_n = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0$, or
- an exponential expression of type $\beta_n = ca^n$.

Moreover there exists a third kind of expression for β which is

- a so-called intermediate expression, that is, an expression that grows faster than a polynomial but slower than an exponential.

Let us introduce some more definitions.

Definition 13 (Exponential growth rate)**.** Let G be the graph generated by S one can define ω, *the exponential growth rate of the pair* (G, S), as

$$\omega(G,S) := \limsup_{n \mapsto \infty} \sqrt[n]{\beta_n(G,S)}.$$

The exponential growth rate detects if β_n is an exponential expression in the sense that $\omega(G,S) > 1$ if and only if β_n is exponential.

Proposition 14. *Let $\beta : \mathbb{N} \longmapsto \mathbb{R}$ be such that $\beta(k+l) \leq \beta(k)\beta(l)$, then the sequence $\{\sqrt[k]{\beta(k)}\}_{k \in \mathbb{N}}$ converges and*

$$\lim_{k \mapsto \infty} \sqrt[k]{\beta(k)} = \inf_{k \geq 1} \sqrt[k]{\beta(k)}.$$

Moreover if λ denote this limit, then $\lambda^k \leq \beta(k) \leq \beta(1)^k$ for all $k \geq 1$.

Proof. Let us define $\alpha(k) = \log \beta(k)$. As $\beta(k+l) \leq \beta(k)\beta(l)$ by hypothesis, one has $\log \beta(k+l) \leq \log(\beta(k)\beta(l))$ and so $\alpha(k+l) \leq \alpha(k) + \alpha(l)$.

Let us choose an integer $n \geq 1$, for all k one can write k as $k = qn + r$ where $q \geq 0$ and $r \in \{1, 2, \ldots, n\}$ since the euclidean decomposition is always possible and in our case if $r_e = 0$ then $r = n$ and $q = q_e - 1$. Hence one has

$$\begin{aligned}
\frac{\alpha(k)}{k} &= \frac{\alpha(qn+r)}{qn+r} \leq \frac{\alpha(qn) + \alpha(r)}{qn+r} = \frac{\alpha(\overbrace{n + \cdots + n}^{q \text{ summands}}) + \alpha(r)}{qn+r} \\
&\leq \frac{q\alpha(n) + \alpha(r)}{qn+r} = \frac{q\alpha(n)}{qn+r} + \frac{\alpha(r)}{qn+r} \leq \frac{q\alpha(n)}{qn} + \frac{\alpha(r)}{k} = \frac{\alpha(n)}{n} + \frac{\alpha(r)}{k}.
\end{aligned}$$

Therefore,

$$\limsup_{k \to \infty} \frac{\alpha(k)}{k} \leq \limsup_{k \to \infty} \frac{\alpha(n)}{n} + \frac{\alpha(r)}{k} = \frac{\alpha(n)}{n}.$$

So $\limsup_{k \to \infty} \frac{\log \beta(k)}{k} = \limsup_{k \to \infty} \log \beta(k)^{1/k} \leq \frac{\log \beta(n)}{n} = \log \beta(n)^{1/n}$, which finally gives

$$\limsup_{k \to \infty} \sqrt[k]{\beta(k)} \leq \sqrt[n]{\beta(n)}.$$

Hence

$$\inf_{n\geq 1} \sqrt[n]{\beta(n)} \leq \lim_{k\to\infty} \sqrt[k]{\beta(k)} \leq \inf_{n\geq 1} \sqrt[n]{\beta(n)} \text{ and the limit } \lim_{k\to\infty} \sqrt[k]{\beta(k)} \text{ exists.}$$

□

Lemma 15. *One can redefine the exponential growth rate as*

$$\omega(G,S) = \lim_{n\mapsto\infty} \sqrt[n]{\beta_n(G,S)}.$$

Proof. If one has $\beta_{n+m} \leq \beta_n \cdot \beta_m$, for all n, m, then the result follows from Proposition **??**. Let us verify that this condition is fulfilled. One knows that

$$B_{n+m} \subseteq \bigcup_{x\in B_n} B_m(x), \text{ where } B_m(x) = \{y \in G \mid d_S(x,y) \leq m\}$$

because for $x \in B_{n+m}$, one can write $x = s_1 \cdots s_{n+m}$ with $s_i \in S \cup S^{-1} \cup \{id\}$. It results that $s_1 \cdots s_n \in B_n$ and

$$d_S(x, s_1 \cdots s_n) = d_S(id, x^{-1}s_1 \cdots s_n) = d_S(id, s_{n+m}^{-1} \cdots s_{n+1}^{-1}) \leq m.$$

Hence $x \in B_m(y)$ with $y \in B_n$, where $B_m(y)$ is the ball of radius m with center y. Therefore

$$\beta_{n+m} = |B_{n+m}| \leq \left| \bigcup_{x\in B_n} B_m(x) \right| = |B_n| \cdot |B_m| = \beta_n \cdot \beta_m.$$

□

Let us briefly retake the examples **??** and calculate their growth rates.

Example 16. 1. *$\mathbb{Z}$, $S = \{1\}$.*
One has see $\beta_n = 2n+1$, so $\omega(\mathbb{Z},S) = 1$.

2. *$\mathbb{Z}^k$, $S = \{e_1, \dots, e_n\}$.*
One established that $\beta_n(\mathbb{Z}^k)$ is a polynomial of degree k, so, as k is fixed, $\omega(\mathbb{Z}^k,S) = \lim_{n\to\infty} \sqrt[n]{\beta_n(\mathbb{Z}^k,S)} = 1$. This result is consistent with the following particular cases.
For $\mathbb{Z}^2$ with $S = \{(1,0),(0,1)\}$, one has shown that $\beta_n = 2n^2+2n+1$, hence $\omega(\mathbb{Z}^2,S) = 1$ and
for $\mathbb{Z}^3$ with $S = \{(1,0,0),(0,1,0),(0,0,1)\}$, one has stated that $\beta_n = \frac{4n^3}{3} + 2n^2 + \frac{8n}{3} + 1$, which implies that $\omega(\mathbb{Z}^3,S) = 1$.

3. *F_2, $S = \{a,b\}$.*
One showed that $\beta_n = 2\cdot 3^n - 1$, so $\omega(\mathbb{F}_2) = 3$.

4. *F_k, $S = \{a_1,\dots,a_k\}$.*
One established that $\beta_n = \frac{k(2k-1)^n - 1}{k-1}$, which leads to $\omega(\mathbb{F}_k) = 2k-1$.

5. $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z} = < a, b \mid a^2 = 1, b^2 = 1 >$.
   One saw that $\beta_n = 1 + 2n$, *hence* $\omega(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}) = 1$.

In these examples one uses the set of canonical generators for the group. The following lemma assures that if one has another finite set of generators, the group will have the same growth type.

Lemma 17. *Let S, T be two finite sets of generators of the group G, then*

$$\omega(G, S) > 1 \Longleftrightarrow \omega(G, T) > 1.$$

Proof. Set $C = \max_{s \in S} d_T(1, S)$, then $B_n(G, S) \subseteq B_{C \cdot n}(G, T)$. It follows directly that $\beta_n(G, S) \leq \beta_{C \cdot n}(G, T)$. Thus

$$\begin{aligned}
\omega(G, S) &= \lim_{n \mapsto \infty} \sqrt[n]{\beta_n(G, S)} \\
&\leq \lim_{n \mapsto \infty} \sqrt[n]{\beta_{C \cdot n}(G, T)} \\
&= \lim_{m \mapsto \infty} \sqrt[m]{\beta_m(G, S)}^{\,C} = \omega(G, T)^C.
\end{aligned}$$

Finally,

$$\omega(G, S) > 1 \Longrightarrow \omega(G, T) > 1.$$

The converse holds by symmetry. □

Definition 18 (Exponential growth)**.** One says that G has *exponential growth* if $\omega(G, S) > 1$ for one and hence any finite set of generators S of G.

One makes a distinction between $\omega(G, S)$ for any finite set S and $\inf_{S \text{ finite}} \omega(G, S)$, this leads to the following definition.

Definition 19 (Uniform exponential growth rate)**.** A group G is said to have *uniform exponential growth rate* if

$$\omega(G) := \inf_S \omega(G, S) > 1,$$

where one take the infimum on all the finite families S generating G.

Remark 20. It has recently been proved that there exist groups with exponential growth but not uniform exponential growth. [1]

The following lemma helps to determine if a group has exponential growth.

Proposition 21. *Let G be a group generated by a family S. If there exists $T \subset S$ such that $< T > = \mathbb{F}_k$, then*

$$\omega(G, S) \geq \omega(< T >, T) = 2k - 1.$$

Consequently, G is of exponential growth.

[1] John S. Wilson, in Inventiones mathematicae, Volume 155, Number 2, February 2004 , pp. 287-303(17)

Proof. It is a consequence of the fact that for a subgroup H of the group G. If H is generated by $R = S \cap H$ where S is the set of generator of G, then

$$\omega(H, R) \leq \omega(G, S).$$

As $R = S \cap H \subseteq S$ and $H \leq G$, one has that the element x in $B_n(H, R)$ are such that $x \in B_n(G, S)$. Hence $\beta_n(H, R) \leq \beta_n(G, S)$ and finally $\omega(H, R) \leq \omega(G, S)$.

□

Let us consider $B(z) = \sum_{n=0}^{\infty} \beta_n(G, S) z^n$ one sees that its convergence radius is

$$\rho_B = \frac{1}{\limsup_{n \mapsto \infty} \sqrt[n]{\beta_n(G, S)}}, \text{ so } \omega(G, S) = \frac{1}{\rho_B}.$$

Moreover, analogously

$$\begin{aligned} C(z) &= \sum_{n=0}^{\infty} \sigma_n(G, S) z^n \\ &= \sum_{n=1}^{\infty} (\beta_n - \beta_{n-1}) z^n + \beta_0 z^0 \\ &= \sum_{n=0}^{\infty} \beta_n z^n - \sum_{n=0}^{\infty} \beta_n z^{n+1} \\ &= B(z) - zB(z) = (1 - z)B(z). \end{aligned}$$

One remarks that if the series $C(z)$ converges, that is $\rho_C < 1$, then $\rho_B = \rho_C$ and $B(z)$ is also converging. It follows that

$$\omega(G, S) = \limsup_{n \mapsto \infty} \sqrt[n]{\beta_n(G, S)} = \frac{1}{\rho_B} = \frac{1}{\rho_C} = \limsup_{n \mapsto \infty} \sqrt[n]{\sigma_n(G, S)}.$$

One the other hand one recalls that ρ_C is also given by calculating

$$\rho_C = \lim_{k \mapsto \infty} \left| \frac{\sigma_k}{\sigma_{k+1}} \right|.$$

Sometimes it will be easier to consider the series $C(z)$.

Example 22. *Let G be $G = \mathbb{Z}_2 * \mathbb{Z}_3 = < a > * < b >$ with $a^2 = id$ and $b^3 = id$.*

1. *$S = \{a, b\}$.*

   Let $C_{i,k}$ denote the set of the elements of G whose length is exactly equal to k and which are ending with i or i^{-1}, where $i \in \{a, b\}$ and let $\sigma_{i,k}$ be the number of elements in $C_{a,k}$. One has that

   - $\sigma_{a,k} = \sigma_{b,k-1}$.

Since for all $x \in C_{a,k}$ one can write $x = s_1 \cdots s_{k-1}a$ and $s_{k-1} \in \{b, b^{-1}\}$, because $a^2 = id$, so $a = a^{-1}$. Therefore $s_1 \cdots s_{k-1} \in C_{b,k-1}$ otherwise x would not be of length equal to k. Hence the number of elements in $C_{a,k}$ is equal to the one in $C_{b,k-1}$.

- $\sigma_{b,k} = 2\sigma_{a,k-1}$.

For all $y \in C_{a,k-1}$ one can write $y = s_1 \cdots s_{k-2}a$. Then $ya \in C_{k-2}$ and $yb, yb^{-1} \in C_{b,k}$, so the number of elements in $C_{a,k-1}$ is two times smaller than the one in $C_{b,k}$.
It follows that

$$\sigma_k = \sigma_{a,k} + \sigma_{b,k} = \sigma_{b,k-1} + 2\sigma_{a,k-1} = 2\sigma_{a,k-2} + 2\sigma_{b,k-2} = 2\sigma_{k-2}.$$

Figure **??** *shows $Cay(G,S)$ with the different colours representing the different σ_k.*

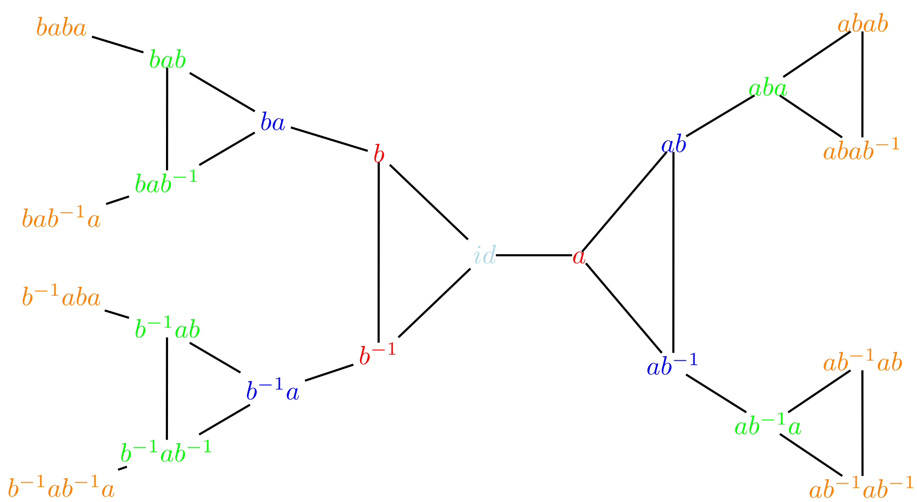


Figure 2.10: $\mathbb{Z}_2 * \mathbb{Z}_3 =< a, b \mid a^2 = 1, b^3 = 1 >$

As σ is always positive, one has

$$\begin{aligned}\omega(G,S) = \frac{1}{\rho_C} &= \frac{1}{\lim\limits_{k \mapsto \infty} \frac{\sigma_k}{\sigma_{k+1}}} \\ &= \lim_{k \mapsto \infty} \frac{\sigma_k}{\sigma_{k-1}} = \lim_{k \mapsto \infty} \frac{2\sigma_{k-2}}{\sigma_{k-1}} \\ &= 2\frac{1}{\omega(G,S)},\end{aligned}$$

so $\omega^2(G,S) = 2$ and finally

$$\omega(G,S) = \sqrt{2}.$$

2. $T = \{a, t\}$, *where* $t = ab$.

One gets so that $at = aab = b$, *which implies that* $(at)^3 = id$. *Moreover,* $tat = at^{-1}a$ *and* $t^{-1}at^{-1} = ata$.

To calculate σ_k, *one knows that the words of length* k *are finishing with* a, t *or* t^{-1}. *Let us define* $\sigma_{a,k}$ *as the number of elements in* $C_{a,k}$, *the set of the words of length* k *which are ending with* a. *As one has seen before any word of length* k *that can be written as* $xtat$ *is the same word as* $xat^{-1}a$ *and similarly* $yt^{-1}at^{-1}$ *is equal to* $yata$. *Therefore one defines* $\sigma_{t,k}$ *as the number of words of length* k *which are ending with* t *or* t^{-1} *but not with* tat *nor* $t^{-1}at^{-1}$ *in order to avoid counting some words twice and* $C_{t,k}$ *the set of those elements. In particular,* $\sigma_k = \sigma_{a,k} + \sigma_{t,k}$.

According to those notations, one has the following relations

- $\sigma_{a,k} = \sigma_{t,k-1}$

Since $a^2 = id$ *every word of length* k *ending with* a *has to be written as* $s_1 \cdots s_{k-1}a$, *where* $s_i \in S \cup S^{-1}$ *and* $s_{k-1} = t$ *or* $s_{k-1} = t^{-1}$. *Moreover there is no possibility that* $s_1 \cdots s_{k-1}$ *ends with* tat *or* $t^{-1}at^{-1}$ *as* $tat = at^{-1}a$ *and* $t^{-1}at^{-1} = ata$ *respectively, which would lead to* $s_1 \cdots s_{k-1}a$ *not being of length* k. *Finally the number of words* $\sigma_{t,k-1}$ *is the same as* $\sigma_{a,k}$.

- $\sigma_{t,k} = \sigma_{t,k-1} + \sigma_{a,k-1} = \sigma_{k-1}$

Let us consider $x \in C_{t,k}$, *where* $C_{t,k}$ *is the set of all the words of length* k, *ending with* t *or* t^{-1}, *but not with* tat *nor* $t^{-1}at^{-1}$. *One can write* x *as* $x_1 = s_1 \cdots s_{k-1}t$ *or* $x_2 = r_1 \cdots r_{k-1}t^{-1}$.

One notes then that $s_{k-1} \in \{a, t\}$, *because if* $s_{k-1} = t^{-1}$ *it would contradict* x *being of length* k, *and* $r_{k-1} \in \{a, t^{-1}\}$ *for the same reasons.*

Let us now consider an element y *in* $C_{t,k-1}$, *then there exists a unique* $s_k = s_{k-1} \in \{t, t^{-1}\}$ *such that* $ys_k \in C_{t,k}$.

In the same way, if one considers $z = c_1 \cdots c_{k-2}a$ *an element in* $C_{a,k-1}$, *it follows that* zt *and* zt^{-1} *are elements whose length is at most equal to* k *and that end with one of* t, t^{-1}. *But, if* $c_{k-2} = t$ *then* zt *is not considered to be an element in* $C_{t,k}$ *as the elements ending in* tat *are excluded from* $C_{t,k}$ *and if* $c_{k-2} = t^{-1}$ *then similarly* zt^{-1} *is not an element in* $C_{t,k}$ *by definition of* $C_{t,k}$. *Hence for each element* z *in* $S_{a,k-1}$, *one can find only one element* c_k *such that* zc_k *is containned in* $C_{t,k}$. *So,*

$$\sigma_{t,k} = \sigma_{a,k-1} + \sigma_{t,k-1}.$$

Hence $\sigma_k = \sigma_{a,k} + \sigma_{t,k} = \sigma_{t,k-1} + \sigma_{k-1} = \sigma_{k-2} + \sigma_{k-1}$.

Figure **??** *shows* $Cay(G, S)$ *with the different colours representing the different* σ.

Since σ *is always positive, one obtains* $\omega(G, S) = \lim_{k \mapsto \infty} \frac{\sigma_k}{\sigma_{k-1}} = \lim_{k \mapsto \infty} \frac{\sigma_{k-1} + \sigma_{k-2}}{\sigma_{k-1}}$, $\omega(G, S) = 1 + \lim_{k \mapsto \infty} \frac{\sigma_{k-2}}{\sigma_{k-1}} = 1 + \frac{1}{\omega(G, S)}$, *thus* $\frac{1}{\omega(G, S)} = \omega(G, S) - 1$. *Finally,*

$$\omega(G, S) = \frac{\sqrt{5} + 1}{2}, \text{ which is the golden ratio.}$$

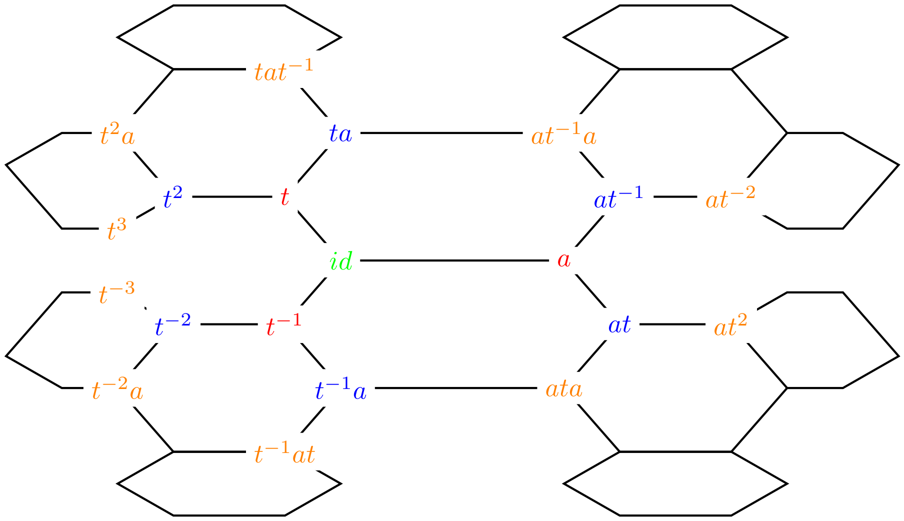


Figure 2.11: $\mathbb{Z}_2 * \mathbb{Z}_3 =< a, t \mid a^2 = 1, (at)^3 = 1 >$

The Cayley graphs of the sets of generators S (dotted) and T (plain lines) are both represented on Figure **??**. *One can see that the group G is really the same.*

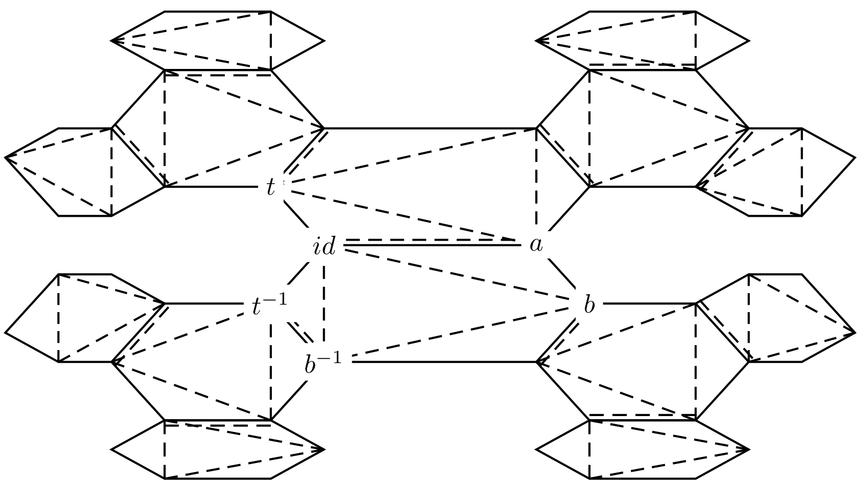


Figure 2.12: $\mathbb{Z}_2 * \mathbb{Z}_3$

Let us consider one more example of calculating $\omega(G,S)$ by counting the words.

Example 23. *Let G be $G = \mathbb{Z}_2 * \mathbb{Z}_4 =< a > * < b >$ with $a^2 = id$ and $b^4 = id$.*

1. *$S = \{a,b\}$.*

   Let $C_{i,k}$ denote the set of the elements of G whose length is exactly equal to k and which are ending with i or i^{-1}, where $i \in \{a,b\}$ and let $\sigma_{i,k}$ be the number of elements in $C_{i,k}$. As $b^4 = 1$, one has that $b^2 = b^{-2}$ and $b^3 = b^{-1}$. Hence in $C_{b,k}$ one will exclude the words ending by b^{-2} or b^3. Let us now prove the following relations

   - $\sigma_{a,k} = \sigma_{b,k-1}$,

   For all $x \in C_{a,k}$ one can write $x = s_1 \cdots s_{k-1}a$. But given that $a^2 = id$ one has $a = a^{-1}$ and therefore $s_1 \cdots s_{k-1} \in C_{b,k-1}$ otherwise x would not be of length equal to k. Hence the number of elements in $C_{a,k}$ is equal to the one in $C_{b,k-1}$.

   - $\sigma_{b,k} = 2\sigma_{a,k-1} + \sigma_{a,k-2}$.

   For all $x \in C_{b,k}$, one can write $x_1 = s_1 \cdots s_{k-1}b$ or $x_2 = r_1 \cdots r_{k-1}b^{-1}$. One sees that $s_{k-1} \in \{a,b\}$ as $s_1 \cdots s_{k-2}b^{-1}b = s_1 \cdots s_{k-2} \notin C_{b,k}$ and $r_{k-1} = a$ because $r_1 \cdots r_{k-2}bb^{-1} = r_1 \cdots r_{k-2} \notin C_{b,k}$ and one has excluded the words ending by b^{-2} from $C_{b,j}$, for all $j \geq 1$.

   On the other hand for all $y \in C_{a,k-1}$, one has that yb and $yb^{-1} \in C_{b,k}$. Moreover, for all $z \in C_{a,k-2}$ one sees that $za \in C_{k-3}$ and $zb, zb^{-1} \in C_{b,k-1}$ which leads to $zbb = zb^{-1}b^{-1} \in C_{b,k}$ and there is no other element in $C_{b,k}$ starting by z. Finally for all $w \in C_{b,k-2}$, the element wa has been considered in $C_{a,k-1}$ and nor wb nor wb^{-1} is a predecessor of some element in $C_{b,k}$. Indeed if one write $w_1 = s_1 \cdots s_{k-3}b$ and $w_2 = s_1 \cdots s_{k-3}b^{-1}$, one has that w_1b^{-1}, $w_2b \in C_{k-3}$ and $w_1b = w_2b^{-1}$ is ending by b^2, so $w_1bb = w_2 \notin C_{b,k}$ and $w_2b^{-1}b^{-1} = w_1 \notin C_{b,k}$.

   Hence,

   $$\sigma_{b,k} = 2\sigma_{a,k-1} + \sigma_{a,k-2}.$$

   One can rewrite $\sigma_{b,k}$ as $\sigma_{b,k} = 2\sigma_{a,k-1}+\sigma_{a,k-2} = \sigma_{a,k-1}+\sigma_{b,k-2}+\sigma_{a,k-2} = \sigma_{a,k-1} + \sigma_{k-2}$.

   Therefore, $\sigma_k = \sigma_{a,k} + \sigma_{b,k} = \sigma_{b,k-1} + \sigma_{a,k-1} + \sigma k - 2 = \sigma_{k-1} + \sigma_{k-2}$.

*Figure **??** shows $Cay(G,S)$ with the different colours representing the different σ and the light colours are the elements of C_a when the dark colours are the ones in C_b.*

*Since σ is always positive, see lemma **??**, one obtains*

$$\begin{aligned}\omega(G,S) &= \lim_{k \mapsto \infty} \frac{\sigma_k}{\sigma_{k-1}} = \lim_{k \mapsto \infty} \frac{\sigma_{k-1} + \sigma_{k-2}}{\sigma_{k-1}} \\ &= 1 + \lim_{k \mapsto \infty} \frac{\sigma_{k-2}}{\sigma_{k-1}} = 1 + \frac{1}{\omega(G,S)}.\end{aligned}$$

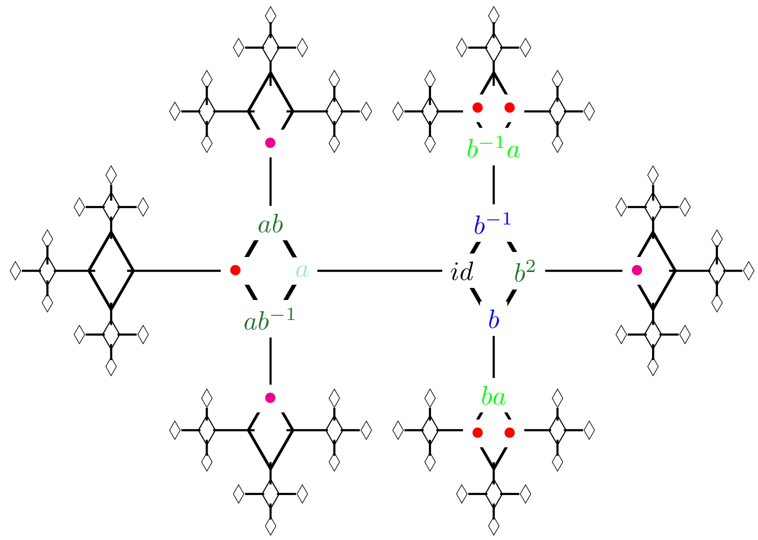


Figure 2.13: $\mathbb{Z}_2 * \mathbb{Z}_4 =< a, b \mid a^2 = 1, b^4 = 1 >$

Finally $\dfrac{1}{\omega(G,S)} = \omega(G,S) - 1$, *that is*

$$\omega(G,S) = \frac{\sqrt{5}+1}{2}, \text{ which is the golden ratio.}$$

2.3 Fundamental groups

Some important examples of groups are fundamental groups. Before giving their definition, let us recall some definitions from topology.

Definition 24 (Path)**.** A *path* or *arc* in a topological space X is a continuous map of some closed interval into X. The images of the end points of the interval are called the *initial* and *terminal* points.

After having defined a path one wants to introduce a topological space in which every path is defined.

Definition 25 (Arcwise connected)**.** One says that the space X is *arcwise connected* if there exists an arc containned in X between each pair of two points of X.

A space X is *locally arcwise connected* if each point has a basic family of arcwise connected neighbourhoods.

It can be shown that a space which is connected and locally arcwise connected is arcwise connected.

One can define a relation of equivalence on the paths.

Definition 26. Let $f, g : [a, b] \to X$ be two paths in X such that

- $f(a) = g(a)$ and
- $f(b) = g(b)$.

One says that these paths are *homotopic relative to their endpoints* and one denotes it by $f \sim g$ if and only if there exists a continuous map $H : [a, b] \times [0, 1] \to X$ such that $\begin{cases} H(t,0) = f(t) \\ H(t,1) = g(t) \end{cases} \forall t \in [a, b]$ and $\begin{cases} H(a,s) = f(a) = g(a) \\ H(b,s) = f(b) = g(b) \end{cases} \forall s \in [0, 1]$.

Claim 27. *Such defined $\sim$ is an equivalence relation.*

Proof. i) $f \sim f$ with $H(t, s) = f(t)$ for all $s \in [0, 1]$.

ii) $f \sim g$ with H, then one can define $G(t, s) = H(t, 1 - s)$ and so

$$\begin{cases} G(t,0) = H(t,1) = g(t) \\ G(t,1) = H(t,0) = f(t) \end{cases} \forall t, \text{ and}$$

$$\begin{cases} G(a,s) = H(a,1-s) = f(a) = g(a) \\ G(b,s) = H(b,1-s) = f(b) = g(b) \end{cases} \forall s, \text{ that is } g \sim f.$$

iii) Let f, g, h be such that $f \sim g$ and $g \sim h$ with G and H respectively, then one defines $F(t,s) = \begin{cases} H(t,2s) & 0 \le s \le \frac{1}{2} \\ G(t,2s-1) & \frac{1}{2} \le s \le 1 \end{cases}$ and so

$$\begin{cases} F(t,0) = H(t,0) = f(t) \\ F(t,\frac{1}{2}) = H(t,1) = G(t,0) = g(t) \\ F(t,1) = G(t,1) = h(t) \end{cases} \forall t, \text{ and}$$

$$\begin{cases} F(a,s) = \begin{cases} H(a,2s) & 0 \le s \le \frac{1}{2} \\ G(a,2s-1) & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} f(a) = g(a) \\ g(a) = h(a) \end{cases} \\ F(b,s) = \begin{cases} H(b,2s) & 0 \le s \le \frac{1}{2} \\ G(b,2s-1) & \frac{1}{2} \le s \le 1 \end{cases} = \begin{cases} f(b) = g(b) \\ g(b) = h(b) \end{cases} \end{cases} \text{so,}$$

$$\begin{cases} F(a,s) = f(a) = h(a) \\ F(b,s) = f(b) = h(b) \end{cases} \forall s, \text{ that is } f \sim h.$$

□

Let us now define some operations on paths.

Definition 28 (Product of two paths)**.** Let $\gamma_0 \colon [0,1] \to X$ and $\gamma_1 \colon [0,1] \to X$ be two paths such that $\gamma_0(1) = \gamma_1(0)$, the *product* of γ_0 and γ_1 is defined as

$$\begin{aligned} \gamma_0 * \gamma_1 \colon [0,1] &\longrightarrow X \\ t &\longmapsto \begin{cases} \gamma_0(2t) & 0 \le t \le \frac{1}{2} \\ \gamma_1(2t-1) & \frac{1}{2} \le t \le 1 \end{cases}. \end{aligned}$$

Definition 29 (Inverse of a path)**.** Let $\gamma\colon [0,1] \to X$ be a path, one defines the *inverse* path of γ as the path with reversed orientation, that is

$$\begin{aligned}\gamma^{-1}\colon [0,1] &\longrightarrow X\\ t &\longmapsto \gamma(1-t).\end{aligned}$$

Claim 30. *Let $e_x\colon [0,1] \to X$, $e_y\colon [0,1] \to X$ and $\gamma\colon [0,1] \to X$ be three paths with $\gamma(0) = x$ and $\gamma(1) = y$ where e_x is the constant path at the point $x \in X$. In the same way is e_y the constant path at point $y \in X$, that is, $e_y(t) = y, \forall t \in [0,1]$, then one has the following*

- $e_x * \gamma \sim \gamma$,
- $\gamma * \gamma^{-1} \sim e_x$ *and*
- $\gamma^{-1} * \gamma \sim e_y$.

Proof. For $e_x * \gamma$, one chooses the function

$$\Gamma(t,s) = \begin{cases} x & 0 \le t \le \frac{s}{2} \\ f(\frac{2t-s}{2-s}) & \frac{s}{2} \le t \le 1 \end{cases}.$$

For $\gamma * \gamma^{-1}$, one chooses the function

$$H(t,s) = \begin{cases} \gamma(2t) & 0 \le t \le \frac{s}{2} \\ \gamma(s) & \frac{s}{2} \le t \le 1 - \frac{s}{2} \\ \gamma(2-2t) & 1 - \frac{s}{2} \le t \le 1 \end{cases}.$$

In the same way one can find a function for $\gamma^{-1} * \gamma$. ☐

Remark 31. Let $f, g, h\colon [0,1] \to X$ such that the products $(f*g)*h$ and $f*(g*h)$ exist, then

$$(f*g)*h \sim f*(g*h).$$

Proof. One chooses the function $H(t,s) = \begin{cases} f(\frac{4t}{1+s}) & 0 \le t \le \frac{s+1}{4} \\ g(4t-1-s) & \frac{s+1}{4} \le t \le \frac{s+2}{4} \\ h(1 - \frac{4(1-t)}{2-s}) & \frac{s+2}{4} \le t \le 1 \end{cases}$. ☐

One sees from the remarks **??** and **??** that if the product of two paths is defined, all the axioms of a group are verified.

Definition 32 (Loop)**.** One defines a *loop* as a closed path, that is $\gamma\colon [0,1] \to X$ is a *loop* if $\gamma(0) = \gamma(1)$.

As the product of two loops is always defined we have a group.

Definition 33 (Fundamental group)**.** One calls $(\Pi_1(X, x_0), *)$ the *fundamental group* of X on base x_0 where X is arcwise connected and $\Pi_1(X, x_0) = \{\gamma\colon [0,1] \to X \mid \gamma(0) = \gamma(1) = x_0\}$ with the relation of equivalence $\sim$.

Theorem 34. *Let X be a connected and locally arcwise connected topological space and let x_0, x_1 be two points in X, then*

$$\Pi_1(X, x_0) \cong \Pi_1(X, x_1).$$

Proof. Let $\gamma\colon [0,1] \to [x_0, x_1]$ be a path. Since X is arcwise connected as it is connected and locally arcwise connected, there exists $\gamma_0 \in [\gamma]$ the class of γ such that $\gamma_0 \subset X$, that is, γ_0 stays in X.

One define then

$$\begin{aligned} u\colon \Pi_1(X, x_0) &\longrightarrow \Pi_1(X, x_1) \\ \alpha &\longmapsto \gamma_0^{-1}\alpha\gamma_0 \end{aligned} \quad \text{and} \quad \begin{aligned} v\colon \Pi_1(X, x_1) &\longrightarrow \Pi_1(X, x_0) \\ \beta &\longmapsto \gamma_0\beta\gamma_0^{-1}. \end{aligned}$$

One sees easily that u and v are homomorphisms. Moreover, $u \circ v = \Pi_1(X, x_0)$ and $v \circ u = \Pi_1(X, x_1)$ which means that u and v are isomorphisms with $u^{-1} = v$. Hence,

$$\Pi_1(X, x_0) \cong \Pi_1(X, x_1).$$

□

It follows from the theorem **??** that the fundamental group of a topological space in point x is isomorphic to the one in point y and there always exists one isomorphism. Hence one makes an abuse in the notation and denotes this group only by $\Pi_1(X)$.

Definition 35 (Homeomorphism)**.** An application between two topological spaces which is bijective, continuous and its reverse function is also continuous is called a *homeomorphism.* If there exists a homeomorphism between two topological spaces they are said to be *homeomorphic.*

Theorem 36. *If two topological spaces X and Y are homeomorphic then their fundamental groups $\Pi_1(X)$ and $\Pi_1(Y)$ are isomorphic.*

Proof. Let $\phi\colon X \to Y$ be a homeomorphism and f_0, $f_1 \in \Pi_1(X)$.

If $f_0 \sim f_1$, then $\phi(f_0) \sim \phi(f_1)$. Let us denote by $\phi_*([\alpha])$ the class of equivalence of $\phi(f_0)$ and $\phi(f_1)$ where $[\alpha]$ denotes the class of f_0, f_1. One sees that if

- $[\alpha]$, $[\beta] \in \Pi_1(X)$ then $\phi_*([\alpha\beta]) = \phi_*([\alpha]\,[\beta]) = \phi_*([\alpha])\phi_*([\beta])$,
- $\forall x \in X$, $\phi_*([e_x]) = \left[e_{\phi(x)}\right]$ and
- $\phi_*(\left[\alpha^{-1}\right]) = (\phi_*([\alpha]))^{-1}$,

  which shows that ϕ_* is a homomorphism.

Moreover as ϕ is a homeomorphism there exists an inverse ϕ^{-1} which is continuous, so that one can define in the same way ϕ_*^{-1}, which has the same properties and

- $\phi_*(\phi_*^{-1}([\alpha])) = [\alpha]$ for all class α in $\Pi_1(X)$,
- $\phi_*^{-1}(\phi_*([\beta])) = [\beta]$ for all class β in $\Pi_1(Y)$.

Hence ϕ_* is an isomorphism and

$$\Pi_1(X) \cong \Pi_1(Y).$$

$\square$

Definition 37 (Homotopy)**.** Two continuous maps $f, g \colon X \to Y$ are *homotopic* if and only if there exists a continuous map $\phi \colon X \times [0,1] \to Y$ such that for $x \in X$ $\begin{cases} \phi(x,0) &= f(x), \\ \phi(x,1) &= g(x). \end{cases}$ One calls a *homotopy* the class of equivalence of f and g.

Remark 38. The relation of equivalence one defined previously on the paths is a homotopy.

Definition 39 (Contractible)**.** A topological space X is *contractible* (to a point) if there exists $x_0 \in X$ and a homotopy $H \colon X \times [0,1] \to X$ such that $\begin{cases} H(x,0) &= x, \\ H(x,1) &= r(x) \end{cases}$ for all $x \in X$ where r is defined by $r \colon X \longrightarrow \{x_0\}$ with $r(x) = x_0$ for all $x \in X$. It means id_X and e_{x_0} are homotopic for a certain $x_0 \in X$.

Theorem 40. *Let X be a contractible topological space then*

$$\Pi_1(X) \cong \{1\}.$$

Proof. The continuous application $r \colon X \longrightarrow \{x_0\}$ which maps all $x \in X$ to $r(x) = x_0$ induces a group homomorphism $r_* \colon \Pi_1(X) \longrightarrow \Pi_1(\{x_0\})$ defined by $r([\alpha]) = e_{x_0}$ for all $[\alpha] \in \Pi_1(X)$. As $\{x_0\} \subset X$ one has $\Pi_1(\{x_0\}) \subset \Pi_1(X)$. Therefore one gets $\Pi_1(X) \cong \Pi_1(\{x_0\})$ and finally as $\Pi_1(\{x_0\}) = \{1\}$,

$$\Pi_1(X) \cong \{1\}.$$

$\square$

Definition 41 (Covering space)**.** A *covering space* of X is a space C together with a continuous surjective map $p \colon C \to X$, such that for every $x \in X$ there exists an open neighbourhood U of x such that $p^{-1}(U)$, the inverse image of U under p , is a disjoint union of open sets in C each of which is mapped homeomorphically onto U by p.

One denotes by (X, C, p) the covering space of C.

Remark 42. Consider the unit circle $\mathbb{S}^1$ in $\mathbb{R}^2$. Then the map $p \colon R \to \mathbb{S}^1$ with $p(t) = (cos(t), sin(t))$ is a cover where each point of $\mathbb{S}^1$ is covered infinitely often, hence $(\mathbb{R}, \mathbb{S}^1, p)$ is a covering space of $\mathbb{S}^1$.

Theorem 43. *Let E and p be a covering space of the topological space B and let $a_0 \in E$ be a point in E and $b_0 = p(a_0) \in B$.*

If $f \colon [0,1] \to B$ is a path such that $f(0) = b_0$ then there exists a unique cover $\widetilde{f} \colon [0,1] \to E$ of f such that $\widetilde{f}(0) = a_0$.

Moreover let $g \colon [0,1] \to B$ be a path such that $g(0) = b_0$, $g(1) = f(1)$ and $g \sim f$ then $\widetilde{g}(1) = \widetilde{f}(1)$ and $\widetilde{g} \sim \widetilde{f}$ where $\widetilde{g}$ denotes the unique cover of g.

Example 44. *One has that*

$$\Pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}.$$

The intuition behind this result is that one should count the number of times that a path wraps around the circle.

Proof. As $\mathbb{S}^1$ is arcwise connected its fundamental group $\Pi_1(\mathbb{S}^1, x_0)$ does not depend on the choice of the point x_0, see Theorem **??**.

Let us consider $x_0 = b_0 = (1, 0)$. Let $(\mathbb{R}, \mathbb{S}^1, p)$ be the covering space of $\mathbb{S}^1$ defined in Remark **??** and choose the point $a_0 = 0$ which respects $p(0) = (1, 0)$.

Let us now define the application

$$\phi \colon \Pi_1(\mathbb{S}^1, b_0) \to \mathbb{Z}.$$

For each loop $f \colon [0, 1] \to \mathbb{S}^1$ of base b_0 let us define $\phi([f]) = \widetilde{f}(1)$ where $\widetilde{f} \colon [0, 1] \to \mathbb{R}$ is the cover of f such that $\widetilde{f}(0) = a_0$. Theorem **??** implies that $\phi([f])$ does not depend on the choice of the representative f of the class $[f]$ and ϕ is well defined.

As $p(\widetilde{f}(1) = f(1) = b_0$, one has $\widetilde{f}(1) \in p^{-1}(\{b_0\}) \in \mathbb{Z}$.

- Let us check that ϕ is injective.

Let f and g be two loops in $\mathbb{S}^1$ of base b_0 such that $\phi([f]) = \phi([g]) = n \in \mathbb{Z}$ and let us show that $f \sim g$. One considers the covers $\widetilde{f}$ and $\widetilde{g}$ such that $\widetilde{f}(0) = \widetilde{g}(0) = a_0$. By definition of ϕ, one gets $\widetilde{f}(1) = \widetilde{g}(1) = n$. As

$$\widetilde{F}(s, t) = (1 - t)\widetilde{f}(s) + t\widetilde{g}(s) \text{ for } s, t \in [0, 1]$$

is a homotopy between both loops, one gets that $p \circ \widetilde{F}$ is a homotopy between $p \circ \widetilde{f} = f$ and $p \circ \widetilde{g} = g$. So

$$[f] = [g]\,.$$

- Let us check that ϕ is surjective.

For $n \in \mathbb{Z}$ let us define the loop in $\mathbb{S}^1$ based on b_0 defined by

$$f_n(s) = (cos(2\pi n s), sin(2\pi n s)) \text{ for } s \in [0, 1]\,.$$

The path $\widetilde{f_n} \colon [0, 1] \to \mathbb{R}$ defined by $\widetilde{f_n}(s) = ns$ for all $s \in [0, 1]$ is a covering of f_n such that $\widetilde{f_n}(0) = a_0$. Finally

$$\phi([f_n]) = \widetilde{f_n}(1) = n.$$

As n was chosen arbitrary ϕ is surjective.

- Let us show that ϕ is a homomorphism.

One will show that for all loops f, g in $\mathbb{S}^1$ based on b_0, one has

$$\phi([f] * [g]) = \phi([f]) + \phi([g]).$$

Let $\widetilde{f}$ and $\widetilde{g}$ be some covering of f and g such that $\widetilde{f}(0) = \widetilde{g}(0) = a_0 = 0$. Let us define $\widetilde{h}(s) = \begin{cases} \widetilde{f}(2s) & \text{if} \quad 0 \leq s \leq \frac{1}{2}, \\ \widetilde{f}(1) + \widetilde{g}(2s-1) & \text{if} \quad \frac{1}{2} \leq s \leq 1. \end{cases}$ As a composition of $\widetilde{f}$ and $\widetilde{g}$, one has that $\widetilde{h}$ is continuous. Moreover

$$p \circ \widetilde{h} = f * g,$$

which implies that $\widetilde{h}$ is a covering of $f * g$, and $p(\widetilde{h}(0)) = p(0) = 0$.

Hence one obtains that

$$\phi([f] * [g]) = \phi([f * g]) = \widetilde{h}(1) = \widetilde{f}(1) + \widetilde{g}(1) = \phi([f]) + \phi([g]).$$

Finally,

$$\phi \text{ is an isomorphism} \quad \text{and} \quad \Pi_1(\mathbb{S}^1, x_0) \cong \mathbb{Z}.$$

□

Theorem 45 (Theorem of Seifert and Van Kampen)**.** *Let U, V be two arcwise connected open subsets of the topological space X such that $X = U \cup V$ and $U \cap V$ is non empty and arcwise connected. Choose a base point $x_0 \in U \cap V$ for all the fundamentals groups, then the diagram of group homomorphism*

$$\begin{tikzcd} & \Pi_1(U) \arrow[dr, "\psi_1"] & \\ \Pi_1(U \cap V) \arrow[ur, "\phi_1"] \arrow[dr, "\phi_2"'] \arrow[rr] & & \Pi_1(X) \\ & \Pi_1(V) \arrow[ur, "\psi_2"'] & \end{tikzcd}$$

is commutative and complete, that is

$$\Pi_1(X) = \Pi_1(U) *_{\Pi_1(U \cap V)} \Pi_1(V).$$

For the definition of product with amalgamation, see Definition 2.

The proof is left to the reader.[2]

2.4 Use of Van Kampen's theorem

One will now use the Van Kampen's theorem to determine some fundamental groups.

Example 46.

$$\Pi_1(\mathbb{S}^2) = \{1\}.$$

Proof. We divide the sphere into two spaces U and V where U is the hole sphere without the North pole and similarly V exclude the South pole. Both spaces are arcwise connected and verify the hypothesis of Theorem **??** so

$$\Pi_1(\mathbb{S}^2) = \Pi_1(U) *_{\Pi_1(U \cap V)} \Pi_1(V).$$

[2]William S. Massey, *A Basic Course in Algebraic Topology*, Chapter IV, Section 2.

One can also see that U and V are contractible to a point, therefore $\Pi_1(U) = \Pi_1(V) = \{1\}$. Moreover one has that $U \cap V$ is a very large band and can be contracted to a circle. Hence,

$$\Pi_1(U \cap V) \cong \Pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

As $\Pi_1(U) = \Pi_1(V)$ we have that $\iota_U(n) = \iota_V(n) = 1$ trivially for all $n \in \mathbb{Z}$. Finally, one gets

$$\Pi_1(\mathbb{S}^2) = \{1\} *_{\mathbb{Z}} \{1\} = \{1\}.$$

□

Remark 47. Using the same kind of space U and V to divide a sphere of higher degree we get that

$$\Pi_1(\mathbb{S}^n) = \{1\} *_{\{1\}} \{1\} = \{1\} \quad \forall n \geq 2.$$

Example 48. *Let us consider the wedge product of two circles connected by a unique point x and denote it as $\mathbb{S}^1_x \vee \mathbb{S}^1_x$, then*

$$\Pi_1(\mathbb{S}^1_x \vee \mathbb{S}^1_x) = \mathbb{F}_2.$$

Proof. Let us divide $\mathbb{S}^1_x \vee \mathbb{S}^1_x$ in U and V with U being the left circle with a part of the right one and V being the right one with part of the left one. Then U and V fulfil the conditions of Theorem **??** and

$$\Pi_1(\mathbb{S}^1_x \vee \mathbb{S}^1_x) = \Pi_1(U) *_{\Pi_1(U \cap V)} \Pi_1(V).$$

One first sees that $U \cap V$ is contractible to x, so

$$\Pi_1(U \cap V) = \{1\},$$

which means that the product with amalgamation in Theorem **??** becomes only a free product.

Otherwise one has that U and V and both contractible in a circle, therefore

$$\Pi_1(U) \cong \Pi_1(\mathbb{S}^1) = \mathbb{Z} \quad \text{and} \quad \Pi_1(V) \cong \Pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

Finally,

$$\Pi_1(\mathbb{S}^1_x \vee \mathbb{S}^1_x) = \Pi_1(U) * \Pi_1(V) = \mathbb{Z} * \mathbb{Z} = \mathbb{F}_2.$$

□

Remark 49. Proceeding in the same way one can generalise this example for more loops and one gets

$$\begin{aligned}\Pi_1(\underbrace{\mathbb{S}^1_x \vee \ldots \vee \mathbb{S}^1_x}_{n \text{ times}}) &\cong \Pi_1(\mathbb{S}^1) * \Pi_1(\underbrace{\mathbb{S}^1_x \vee \ldots \vee \mathbb{S}^1_x}_{n-1 \text{ times}}) = \mathbb{Z} * \Pi_1(\underbrace{\mathbb{S}^1_x \vee \ldots \vee \mathbb{S}^1_x}_{n-1 \text{ times}}) \\ &= \cdots = \underbrace{\mathbb{Z} * \cdots * \mathbb{Z}}_{n \text{ times}} = \mathbb{F}_n.\end{aligned}$$

Example 50. *Let us denote the torus as $\mathbb{T}_1$, then*

$$\Pi_1(\mathbb{T}_1) = \mathbb{Z}^2.$$

Proof. Let us cut a disk on the surface of the torus and call this disk V and let U be the rest of the torus with a smaller hole so that $U \cap V$ is non empty. Then U and V fulfil the condition of Theorem **??** and

$$\Pi_1(\mathbb{T}_1) = \Pi_1(U) *_{\Pi_1(U \cap V)} \Pi_1(V).$$

First one remarks that the disk V is contractible, therefore

$$\Pi_1(V) = \{1\}.$$

Then one sees that $U \cap V$ is a band and therefore is contractible to a circle, so

$$\Pi_1(U \cap V) \cong \Pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

It is slightly more difficult to understand what is the fundamental group of U. If one uses the plane representation of the torus where one indicate which sides are connected which each other and in which direction, one can contract U to the edges of the figure. That is U can be contracted to two circles connected by a line. This space can be even more contracted to the wedge product of two circles. Hence,

$$\Pi_1(U) \cong \Pi_1(\mathbb{S}^1_x \vee \mathbb{S}^1_x) = \mathbb{F}_2.$$

By our choice of U and V the homomorphisms ι_U and ι_V are given by

$$\iota_U(1) = aba^{-1}b^{-1}, \text{ where } \mathbb{F}_2 =< a, b > \quad \text{and} \quad \iota_V(n) = 1, \forall n \in \mathbb{Z}$$

then $\iota_U(1) = \iota_V(1)$ if and only if $aba^{-1}b^{-1} = 1$. Hence one gets

$$\Pi_1(\mathbb{T}_1) = \mathbb{F}_2 *_{\mathbb{Z}} \{1\} =< a, b \mid aba^{-1}b^{-1} = 1 >= \mathbb{Z}^2.$$

□

Example 51. *Let us denote the surface with two holes as Σ_2, then*

$$\Pi_1(\Sigma_2) =< a_1, a_2, b_1, b_2 \mid [a_1, b_1]\,[a_2, b_2] = 1 > .$$

Proof. Let us divide the torus with two holes in two parts that are slightly bigger than a torus. So both parts cut them and they are similar to a torus of which a disk is taken away. Then let us call them U and V.

As $U \cap V$ is non empty and they are arcwise connected, those U and V fulfil the condition of Theorem **??** and

$$\Pi_1(\Sigma_2) = \Pi_1(U) *_{\Pi_1(U \cap V)} \Pi_1(V).$$

First one sees that the intersection $U \cap V$ can be assimilated as a cylinder. Hence as the cylinder is contractible into a circle one has

$$\Pi_1(U \cap V) \cong \Pi_1(\mathbb{S}^1) = \mathbb{Z}.$$

One has seen in Example **??** that a torus without a disk can be contracted to the wedge product of two circles. Therefore

$$\Pi_1(U) = \Pi_1(V) \cong \Pi_1(\mathbb{S}^1_x \vee \mathbb{S}^1_x) = \mathbb{F}_2.$$

By our choice of U and V the homomorphisms ι_U and ι_V are given by

$$\iota_U(1) = a_1 b_1 a_1^{-1} b_1^{-1}, \text{ where } \mathbb{F}_2 =< a_1, b_1 > \quad \text{and}$$
$$\iota_V(1) = b_2 a_2 b_2^{-1} a_2^{-1}, \text{ where } \mathbb{F}_2 =< a_2, b_2 > .$$

Then $\iota_U(1) = \iota_V(1)$ if and only if $a_1 b_1 a_1^{-1} b_1^{-1} = b_2 a_2 b_2^{-1} a_2^{-1}$. Hence one gets

$$\begin{aligned}\Pi_1(\Sigma_2) = \mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2 &=< a_1, a_2, b_1, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} = b_2 a_2 b_2^{-1} a_2^{-1} > \\ &=< a_1, a_2, b_1, b_2 \mid a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} = 1 > \\ &=< a_1, a_2, b_1, b_2 \mid [a_1, b_1]\,[a_2, b_2] = 1 > .\end{aligned}$$

□

Remark 52. Proceeding in the same way one can generalise this example for a surface with n holes and one gets

$$\begin{aligned}\Pi_1(\Sigma_n) &\cong \Pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) *_{\mathbb{Z}} \Pi_1(\Sigma_{n-1}) = \mathbb{F}_2 *_{\mathbb{Z}} \Pi_1(\Sigma_{n-1}) = \mathbb{F}_2 *_{\mathbb{Z}} \mathbb{F}_2 *_{\mathbb{Z}} \Pi_1(\Sigma_{n-2}) \\ &= \cdots = \underbrace{\mathbb{F}_2 *_{\mathbb{Z}} \cdots *_{\mathbb{Z}} \mathbb{F}_2}_{n \text{ times}} =< a_1, b_1, \ldots, a_n, b_n \mid \prod_{i=1}^{n} [a_i, b_i] = 1 > .\end{aligned}$$

2.5 Some growth rates of fundamental groups

One has shown previously that $\Pi_1(X)$ is a group, so one can calculate its growth rate. Let us reconsider the examples we have seen in Part **??**.

First one can easily compute the following growth rates.

- $\omega(\Pi_1(\mathbb{S}^1)) = 0$ as one has shown in Example **??** that $\Pi_1(\mathbb{S}^1) = \{1\}$, so $\beta_n = 0$ for all $n > 0$.
- $\omega(\Pi_1(\mathbb{S}^1 \vee \mathbb{S}^1)) = 3$, because $\Pi_1(\mathbb{S}^1 \vee \mathbb{S}^1) = \mathbb{F}_2$, which has has been shown in Example **??**.
- $\omega(\Pi_1(\mathbb{T}_1)) = 1$ since one has seen in Example **??** that $\Pi_1(\mathbb{T}_1) = \mathbb{Z}^2$.

One now wants to estimate the growth rate of $\Pi_1(\Sigma_2)$, but one has seen in Example **??** that

$$\Pi_1(\Sigma_2) =< a_1, a_2, b_1, b_2 \mid [a_1, b_1]\,[a_2, b_2] = 1 >,$$

which is not a group for which we already computed β_n.

Let us fist consider $\Pi_1(\Sigma_2)$ generated by $S = \{a_1, b_1, a_2, b_2\}$. The subset T of S given by $T = \{a_1, b_1, a_2\}$ is such that $T \cong \mathbb{F}_3$ since there is no relation between a_1, b_1 and a_2. The proposition **??** implies that

$$\omega(\Pi_1(\Sigma_2), S) \geq \omega(\mathbb{F}_3, T) = 2 \cdot 3 - 1 = 5 > 1.$$

So $\Pi_1(\Sigma_2)$ has exponential growth.

Moreover, there exists a homomorphism

$$c\colon \Pi_1(\Sigma_k) \to (\Pi_1(\Sigma_k))^{ab} = \Pi_1(\Sigma_k)/\,[\Pi_1(\Sigma_k), \Pi_1(\Sigma_k)]$$

which sends $\Pi_1(\Sigma_k)$ on its abelianized group $(\Pi_1(\sigma_k))^{ab} = \mathbb{Z}^{2k}$. The image of any set S generating $\Pi_1(\Sigma_k)$ is then a generating set of $\mathbb{Z}^{2k}$. Thus there exists R a subset of S of $2k$ elements which are generating a subgroup of finite index $c(R)$ in $\mathbb{Z}^{2k}$. Let us define R_0 as R without one (arbitrary) element in R, then R_0 generates a subgroup $< R_0 >$ of finite index in $\Pi_1(\Sigma_k)$. Since $< R_0 >$ is the fundamental group of a non-compact surface it is free and its rank is exactly $2k-1$ because its abelianization is isomorphic to $\mathbb{Z}^{2k-1}$. Hence,

$$\omega(\Pi_1(\Sigma_k)) \geq \omega(< R_0 >, R_0) = 2(2k-1) - 1 = 4k - 3.$$

Therefore $\omega(\Pi_1(\Sigma_k)) \geq 4k - 3 > 1$ for every generating family, which implies that $\omega(\Pi_1(\Sigma_k))$ has uniformly exponential growth for all $k \geq 2$.

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